

Singular order parameter interaction at nematic quantum critical point in two dimensional electron systems

Stephan C. Thier^{1,2,*} and Walter Metzner²

¹*Universität Stuttgart, Fachbereich Physik, D-70550 Stuttgart, Germany*

²*Max-Planck-Institute for Solid State Research, D-70569 Stuttgart, Germany*

(Dated: March 22, 2012)

We analyze the infrared behavior of effective N -point interactions between order parameter fluctuations for nematic and other quantum critical electron systems with a scalar order parameter in two dimensions. The interactions exhibit a singular momentum and energy dependence and thus cannot be represented by local vertices. They diverge for all $N \geq 4$ in a collinear low-energy limit, where energy variables scale to zero faster than momenta, and momenta become increasingly collinear. The degree of divergence is not reduced by any cancellations and renders all N -point interactions marginal. A truncation of the order parameter action at quartic or any other finite order is therefore not justified. The same conclusion can be drawn for the effective action describing fermions coupled to a $U(1)$ gauge field in two dimensions.

PACS numbers: 05.30.Rt, 71.10.Hf, 71.27.+a

I. INTRODUCTION

Numerous interacting electron systems undergo a quantum phase transition¹ between ground states with different symmetries, which can be tuned by a non-thermal control parameter such as doping, pressure, or a magnetic field. In the vicinity of a continuous transition electronic excitations are strongly scattered by critical order parameter fluctuations, such that Fermi liquid theory breaks down.^{2,3} Quantum critical fluctuations near a quantum critical point (QCP) are therefore frequently invoked as a mechanism for non-Fermi liquid behavior in strongly correlated electron compounds.

Quantum criticality in metallic electron systems is traditionally described by an effective order parameter theory which was pioneered by Hertz⁴ and extended to finite temperatures by Millis.⁵ In that approach an order parameter field ϕ is introduced via a Hubbard-

Stratonovich decoupling of the electron-electron interaction, and the electronic variables are subsequently integrated out. The resulting effective action $S[\phi]$ for the order parameter is truncated at quartic order and analyzed by standard scaling techniques.

However, several studies revealed that the Hertz-Millis approach may fail, especially in low-dimensional systems.^{3,6} Since electronic excitations in a metal are gapless, integrating out the electrons may lead to singular interactions between the order parameter fluctuations which cannot be approximated by a local quartic term. The nature of the problem and essential aspects of its solution were presented first for disordered ferromagnets by Kirkpatrick and Belitz.⁷ For clean ferromagnets, Belitz et al.⁸ showed that Hertz-Millis theory breaks down, and no continuous quantum phase transition can exist, in any dimension $d \leq 3$; the transition is generically of first order.⁹ The Hertz-Millis approach was also shown to be invalid for the quantum antiferromagnetic transition in two dimensions.¹⁰⁻¹² In that case a continuous transition survives, but the QCP becomes non-Gaussian.

In this article we analyze the validity of the Hertz-Millis approach to quantum criticality for two-dimensional systems with singular *forward scattering* of electrons, in particular systems exhibiting a quantum phase transition driven by forward scattering in the charge channel. The most prominent such transition is the electronic *nematic*, in which an orientation symmetry is spontaneously broken, while translation and spin-rotation invariance remain unaffected.¹³ The problem of quantum critical points with singular forward scattering is formally similar to the problem of non-relativistic fermions coupled to a $U(1)$ gauge field, which was studied intensively in the 1990s.^{14,15}

Perturbation theory for the electronic self-energy at the nematic QCP yields a non-Fermi liquid contribution proportional to $\omega^{2/3}$ already at the lowest order in a loop expansion.^{16,17} The same behavior was found earlier for fermions coupled to a $U(1)$ gauge field.¹⁸ It was commonly believed that the power-law with an exponent $\frac{2}{3}$ is not modified by higher order contributions. Furthermore, calculations in the gauge field context suggested that the simple form of the (bosonic) fluctuation propagator obtained in lowest order (RPA) remains unaffected by higher order terms.¹⁹ In a remarkable recent paper Metlitski and Sachdev²⁰ formulated a scaling theory of the nematic QCP and related problems, treating the electrons and order parameter fluctuations on equal footing. In a renormalization group calculation they found a logarithmic divergence at three-loop order pointing at a correction of the $\omega^{2/3}$ law for the electronic self-energy. However, no qualitative correction was found for the fluc-

tuation propagator, up to three-loop order.^{21,22} This is in stark contrast to the case of an antiferromagnetic QCP in two dimensions, where the fluctuation propagator is substantially renormalized compared to the RPA form.^{10,12} A clarification of the properties of the nematic QCP beyond three-loop order is still lacking.

The robustness of the fluctuation propagator at the nematic QCP seems to indicate that interactions of the order parameter fluctuations are irrelevant such that the QCP is Gaussian, in agreement with the expectations from Hertz-Millis theory. It is therefore worthwhile to analyze the interaction terms in the effective action $S[\phi]$ obtained after integrating out the electrons. The N -point interactions are given by fermionic loops with N vertices. To obtain the scaling behavior of such loops is non-trivial, because the most naive power-counting is easily invalidated by cancellations.^{23,24} In this paper we compute the exact scaling behavior of the N -point interactions for the nematic QCP and related systems. We find that the interactions are *marginal* and *non-local* for all $N \geq 3$. Hence, replacing them by a local ϕ^4 interaction is not justified.

The paper is structured as follows. In Sec. II we introduce the effective actions to be analyzed, and we define the N -point loops describing the interaction terms. In Sec. III we explain the special role of fluctuations with collinear momenta, which motivates the definition of the collinear low-energy scaling limit. Secs. IV-VI are dedicated to the analysis of the N -point loops. After reviewing exact formulae from the literature (Sec. IV), we derive explicit expressions for N -point loops in the collinear low-energy scaling limit. Using the scaling behavior of these loops, we perform the power counting of N -point order parameter interactions in Sec. VII. We finally conclude in Sec. VIII.

II. EFFECTIVE ACTION AND N-POINT LOOPS

We consider an interacting Fermi system which undergoes a continuous quantum phase transition with a scalar order parameter of the form

$$O = \sum_{\sigma} \int \frac{d^2k}{(2\pi)^2} d_{\sigma}(\mathbf{k}) c_{\sigma}^{\dagger}(\mathbf{k}) c_{\sigma}(\mathbf{k}), \quad (1)$$

where $c_{\sigma}^{\dagger}(\mathbf{k})$ and $c_{\sigma}(\mathbf{k})$ are the usual fermionic creation and annihilation operators. For a charge nematic¹³ the form factor $d_{\sigma}(\mathbf{k})$ is spin symmetric and has a \mathbf{k} -dependence with d -wave symmetry, such as $d_{\sigma}(\mathbf{k}) = \cos k_x - \cos k_y$. A spin-antisymmetric form factor may

describe an Ising ferromagnet or an Ising spin nematic.

Decoupling the fermionic interaction by introducing an order parameter field ϕ via a Hubbard-Stratonovich transformation, and integrating out the fermionic variables,⁴ one obtains an effective action

$$S[\phi] = \frac{1}{2} \int_q g^{-1} \phi(q) \phi(-q) + \sum_{N=2}^{\infty} \frac{(-1)^N}{N} \int_{q_1, \dots, q_N} \delta(q_1 + \dots + q_N) \Pi_{d,N}(q_1, \dots, q_N) \phi(q_1) \dots \phi(q_N), \quad (2)$$

where $g > 0$ is the fermionic coupling constant, and

$$\Pi_{d,N}(q_1, \dots, q_N) = \sum_{\sigma} \int_k \prod_{j=1}^N [d_{\sigma}(\mathbf{k} - \mathbf{p}_j - \mathbf{q}_j/2) G_0(k - p_j)] . \quad (3)$$

Here and in the following we use 3-vectors collecting imaginary frequency and two-dimensional momentum variables, for example $k = (k_0, \mathbf{k})$, and \int_k as a short-hand notation for $\int \frac{dk_0}{2\pi} \frac{d^2 k}{(2\pi)^2}$. The variables p_j and q_j are related by

$$\begin{aligned} q_j &= p_{j+1} - p_j \quad \text{for } j = 1, \dots, N-1 \\ q_N &= p_1 - p_N . \end{aligned} \quad (4)$$

Note that $q_1 + \dots + q_N = 0$ due to energy and momentum conservation. The bare propagator has the form $G_0(k) = [ik_0 - \epsilon(\mathbf{k}) + \mu]^{-1}$, where $\epsilon(\mathbf{k})$ is the dispersion relation of the non-interacting particles. $\Pi_{d,N}(q_1, \dots, q_N)$ can be represented graphically as a fermion loop with N lines corresponding to G_0 and N vertices with form factors $d_{\sigma}(\mathbf{k})$, as shown in Fig. 1. For spin-antisymmetric form factors, $\Pi_{d,N}$ vanishes for odd N .

For fermions coupled to a $U(1)$ gauge field, integrating out the fermions leads to a similar effective action $S[\phi]$, where ϕ is the transverse component of the gauge field (in Coulomb gauge). The bosonic N -point functions are then essentially given by a loop with transverse current vertices

$$\Pi_{t,N}(q_1, \dots, q_N) = 2 \int_k \prod_{j=1}^N [\hat{\mathbf{q}}_{j\perp} \cdot \mathbf{v}(\mathbf{k} - \mathbf{p}_j - \mathbf{q}_j/2) G_0(k - p_j)] , \quad (5)$$

where $\mathbf{v}(\mathbf{k}) = \nabla \epsilon(\mathbf{k})$, and $\hat{\mathbf{q}}_{\perp}$ is the unit vector obtained by rotating $\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$ by $\pi/2$, that is, $\hat{\mathbf{q}}_{\perp} = (-\hat{q}_y, \hat{q}_x)$. In addition there are contributions from the “diamagnetic” term (of the form $\phi^2 \bar{\psi} \psi$) in the underlying fermionic action, which are however less singular, since they involve a smaller number of propagators (less than N).

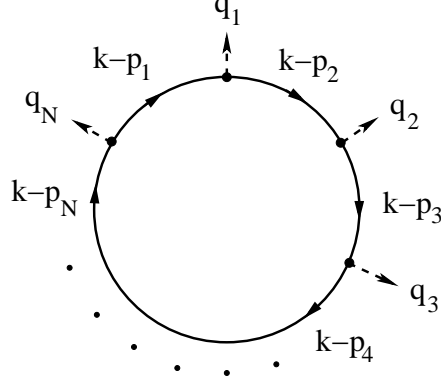


FIG. 1: Graphical representation of $\Pi_{d,N}$ with momentum variables as in Eq. (3).

The N -point contribution to $S[\phi]$ in Eq. (2) is symmetric under any permutation of q_1, \dots, q_N . Hence, one can replace $\Pi_{d,N}$ by the symmetrized N -point loop

$$\Pi_{d,N}^S(q_1, \dots, q_N) = \frac{1}{N!} \sum_P \Pi_{d,N}(q_{P1}, \dots, q_{PN}), \quad (6)$$

where the sum collects all permutations P of $1, \dots, N$. Non-symmetric contributions to $\Pi_{d,N}$ do not contribute to the integral in Eq. (2). For the gauge field problem one defines $\Pi_{t,N}^S$ analogously. Substantial cancellations may occur in the sum over permutations.^{23,24}

III. COLLINEAR LOW-ENERGY LIMIT

The quadratic part of $S[\phi]$ is determined by a constant and the 2-point loop or “bubble”

$$\Pi_d(q) = \Pi_{d,2}(q, -q) = \sum_{\sigma} \int_k d_{\sigma}^2(\mathbf{k}) G_0(k - q/2) G_0(k + q/2). \quad (7)$$

For small \mathbf{q} and small $|q_0|/|\mathbf{q}|$, it has an expansion of the form^{4,25}

$$\Pi_d(q) = -N_d + \chi_d \mathbf{q}^2 + \gamma_d \frac{|q_0|}{|\mathbf{q}|} + \dots, \quad (8)$$

where $N_d = \sum_{\sigma} \int \frac{d^2 k}{(2\pi)^2} d_{\sigma}^2(\mathbf{k}) \delta[\epsilon(\mathbf{k}) - \mu]$ is a weighted density of states, and χ_d and γ_d are two other constants. At the quantum critical point one has $g^{-1} - N_d = 0$ such that the quadratic part of the action vanishes for $\mathbf{q} \rightarrow 0$ and $q_0/|\mathbf{q}| \rightarrow 0$. For the gauge problem, the constant N_t from the static limit of $\Pi_t(q)$ cancels generically against a tadpole contribution,^{14,15} such that the theory is always critical. In both cases, the Gaussian part of the action has thus the asymptotic form

$$S_0[\phi] = \frac{1}{2} \int_q \left(\chi \mathbf{q}^2 + \gamma \frac{|q_0|}{|\mathbf{q}|} \right) \phi(q) \phi(-q), \quad (9)$$

corresponding to a bare propagator

$$D_0(q) = \frac{1}{\chi \mathbf{q}^2 + \gamma \frac{|q_0|}{|\mathbf{q}|}}. \quad (10)$$

$D_0(q)$ diverges in the limit $\mathbf{q} \rightarrow 0$ and $q_0/|\mathbf{q}| \rightarrow 0$. The two terms in the denominator of $D_0(q)$ vanish at the same pace for $\mathbf{q} \rightarrow 0$ if $q_0 \propto |\mathbf{q}|^3$. Therefore, the bare dynamical scaling exponent is $z = 3$. To assess the size of the interaction terms in $S[\phi]$ one thus has to study the N -point loops in a low-energy limit with $q_{j0} \propto |\mathbf{q}_j|^3$. Naively one would expect that this corresponds to the *static* limit, where $q_{j0} \rightarrow 0$ before $\mathbf{q}_j \rightarrow 0$. Hertz and Klenin²⁶ showed that an N -point loop converges to the $(N - 2)$ -th derivative of the density of states with respect to the Fermi energy in the static limit. In our case, with a form factor $d_\sigma(\mathbf{k})$, their result generalizes to

$$\lim_{\mathbf{q}_j \rightarrow 0} \lim_{q_{j0} \rightarrow 0} \Pi_{d,N}(q_1, \dots, q_N) = \frac{(-1)^{N-1}}{(N-1)!} \frac{\partial^{N-2}}{\partial \mu^{N-2}} \sum_\sigma \int \frac{d^2 k}{(2\pi)^2} d_\sigma^N(\mathbf{k}) \delta[\epsilon(\mathbf{k}) - \mu]. \quad (11)$$

Except for special cases where the chemical potential lies at a van Hove singularity, this expression is *finite*. Note that the right hand side of Eq. (11) is independent of q_1, \dots, q_N and hence already symmetrized. Approximating the bosonic N -point interactions by finite local interactions thus seems adequate. Standard power counting then implies that all interactions with $N \geq 4$ are increasingly (with higher N) irrelevant. Hence, the Hertz-Millis truncation seems justified for QCPs with singular forward scattering, even in two dimensions. The static limit of the 3-point loop and all other N -point loops with odd N as given by Eq. (11) vanishes even in the case of a charge nematic, due to the antisymmetry of $d_\sigma(\mathbf{k})$ under $\pi/2$ rotations of \mathbf{k} .

One arrives at a similar conclusion for the gauge field problem. In that case the static limit of $\Pi_{t,N}$ is also generically finite. Odd N -point interactions vanish due to the antisymmetry of $\mathbf{v}(\mathbf{k})$ and even N -point interactions appear to be irrelevant for any $N \geq 4$. A Gaussian fixed point thus seems natural.

However, there is a flaw in the above argument. Eq. (11) has been derived by setting $q_{j0} = 0$ before the momenta \mathbf{q}_j tend to zero. It is not guaranteed that this captures the low-energy limit $\mathbf{q}_j \rightarrow 0$ and $q_{j0}/|\mathbf{q}_j| \rightarrow 0$ in general. Indeed, a simple estimate indicates that the N -point loop is of order $q_{j0}/|\mathbf{q}_j|^{N-1}$ for small non-collinear momenta \mathbf{q}_j and small $q_{j0}/|\mathbf{q}_j|$.²⁰ Although this behavior is increasingly singular for larger N , the corresponding order parameter interactions remain irrelevant, since the singularity is not strong enough.²⁰

However, an even stronger singularity is obtained in a special low-energy limit in which the momenta $\mathbf{q}_1, \dots, \mathbf{q}_N$ become *collinear*. The crucial role of coupled fluctuations with collinear momenta was highlighted very clearly by Metlitski and Sachdev.²⁰ In perturbative one-loop calculations of the fermionic self-energy $\Sigma(\mathbf{k}_F, \omega)$ at a certain point \mathbf{k}_F on the Fermi surface, it was found already some time ago that the dominant contributions involve only fermionic states in the momentum region near \mathbf{k}_F and $-\mathbf{k}_F$, with momentum transfers \mathbf{q} almost *tangential* to the Fermi surface in those points. See, for example, Ref. 18 for an early calculation in the gauge field context, and Ref. 17 for a corresponding calculation at the nematic QCP. This remains true for higher order contributions,²⁰ so that all fermionic momenta are close to \mathbf{k}_F and $-\mathbf{k}_F$ and momentum transfers are almost tangential to the Fermi surface in these points, which implies that they are mutually almost collinear.

Choosing a coordinate system in momentum space in such a way that the normal vector to the Fermi surface at \mathbf{k}_F points in x -direction, the proper scaling limit describing the low-energy behavior is given by $k_0 \mapsto \lambda^3 k_0$, $k_x \mapsto \lambda^2 k_x$, and $k_y \mapsto \lambda k_y$ with $\lambda \rightarrow 0$, where (k_x, k_y) is measured relative to \mathbf{k}_F .^{20,27,28} For the momentum and energy transfers q_j this implies

$$q_{j0} \mapsto \lambda^3 q_{j0}, \quad q_{jx} \mapsto \lambda^2 q_{jx}, \quad q_{jy} \mapsto \lambda q_{jy} \quad (12)$$

with $\lambda \rightarrow 0$. In this *collinear low-energy limit* the momentum transfers \mathbf{q}_j become increasingly collinear (pointing in y -direction). The behavior of the N -point interactions given by $\Pi_{d,N}(q_1, \dots, q_N)$ and $\Pi_{t,N}(q_1, \dots, q_N)$ in the collinear low-energy limit has not yet been studied systematically. In particular, it has not yet been analyzed whether cancellations suppress their value below the naive power counting estimate. To clarify this issue is the main purpose of our article.

As mentioned above, the dominant contributions are due to momenta \mathbf{k} close to those points $\pm\mathbf{k}_F$ on the Fermi surface at which the momentum transfers \mathbf{q}_j are tangent. In the defining equation (3) for $\Pi_{d,N}$ we can therefore replace the form factors $d_\sigma(\mathbf{k})$ by $d_\sigma(\pm\mathbf{k}_F)$. Assuming $d_\sigma(-\mathbf{k}_F) = d_\sigma(\mathbf{k}_F)$, which is satisfied in all cases of interest, we then obtain

$$\Pi_{d,N}(q_1, \dots, q_N) \rightarrow \sum_{\sigma} d_{\sigma}^N(\mathbf{k}_F) \Pi_N(q_1, \dots, q_N) \quad (13)$$

in the collinear low-energy limit, where

$$\Pi_N(q_1, \dots, q_N) = I_N(p_1, \dots, p_N) = \int_k \prod_{j=1}^N G_0(k - p_j) \quad (14)$$

is the N -point loop for spinless fermions with unit vertices. Furthermore, the dispersion $\epsilon(\mathbf{k})$ enters only via the Fermi velocity v_F and the Fermi surface curvature in $\pm\mathbf{k}_F$. Both are assumed to be finite, which is the generic case. We choose units such that v_F and the curvature radius ρ_F are both one, and we realize these parameters by using a simple parabolic dispersion relation $\epsilon(\mathbf{k}) = \mathbf{k}^2/2$ and setting $k_F = 1$. Relating $\Pi_{d,N}$ to Π_N with a quadratic dispersion enables us to exploit exact results for Π_N which are already available (see below). For the gauge field problem the vertices are antisymmetric under reflections, since $\mathbf{v}(-\mathbf{k}) = -\mathbf{v}(\mathbf{k})$. Hence, $\Pi_{t,N}$ can be reduced to Π_N in the collinear low-energy limit only for *even* N :

$$\Pi_{t,N}(q_1, \dots, q_N) \rightarrow 2 \prod_{j=1}^N \hat{\mathbf{q}}_{j\perp} \cdot \mathbf{v}(\mathbf{k}_F) \Pi_N(q_1, \dots, q_N). \quad (15)$$

For odd N , contributions from \mathbf{k} near \mathbf{k}_F and $-\mathbf{k}_F$ contribute with opposite sign and can therefore not be written in terms of Π_N . However, the results obtained for even N suffice to show that the effective action involves non-local marginal interaction of arbitrarily high order.

The symmetrized N -point loop

$$\Pi_N^S(q_1, \dots, q_N) = \frac{1}{N!} \sum_P \Pi_N(q_{P1}, \dots, q_{PN}) \quad (16)$$

describes the dynamical N -point *density correlations* of a Fermi gas. In the following sections we will derive its scaling behavior in the collinear low-energy limit for arbitrary N .

IV. EXACT FORMULAE FOR N-POINT DENSITY LOOP

Our analysis of the scaling behavior of Π_N and Π_N^S is based on exact expressions derived by Feldman et al.²⁹ and their elaboration by Neumayr and Metzner.²³ They are valid for a parabolic dispersion relation. In this section we summarize these expressions, assuming specifically $\epsilon(\mathbf{k}) = \mathbf{k}^2/2$ and $k_F = 1$. Obviously one may restore an arbitrary mass and k_F at will. We use the parametrization $I_N(p_1, \dots, p_N)$ with momenta p_j linearly related to the q_j , as described in Sec. II, see also Fig. 1. The following expressions are applicable only for non-collinear momenta. Nevertheless, they can be used to study the scaling behavior in a limit where they become increasingly collinear upon reducing λ .

The N -point loop can be expressed as a linear combination of 3-point loops with rational coefficients:^{23,29,30}

$$I_N(p_1, \dots, p_N) = \sum_{1 \leq i < j < k \leq N} \left[\prod_{\substack{\nu=1 \\ \nu \neq i, j, k}}^N f_{i\nu}(\mathbf{d}^{ijk}) \right]^{-1} I_3(p_i, p_j, p_k), \quad (17)$$

where

$$f_{i\nu}(\mathbf{d}^{ijk}) = \frac{1}{2}(\mathbf{p}_i^2 - \mathbf{p}_\nu^2) + i(p_{i0} - p_{\nu 0}) + \left\{ \left[\frac{1}{2}(\mathbf{p}_k^2 - \mathbf{p}_i^2) + i(p_{k0} - p_{i0}) \right] \frac{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_\nu - \mathbf{p}_i)}{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} + j \leftrightarrow k \right\}. \quad (18)$$

The determinants of two momenta are defined as $\det(\mathbf{p}, \mathbf{p}') = \det \begin{pmatrix} p_x & p'_x \\ p_y & p'_y \end{pmatrix}$.

For the 3-point loop, Feldman et al.²⁹ have obtained the expression

$$I_3(p_1, p_2, p_3) = \frac{1}{2\pi i \det(\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1)} \sum_{\substack{i, j=1 \\ i \neq j}}^3 s_{ij} t_{ij} \quad (19)$$

where $s_{12} = s_{23} = s_{31} = 1$, $s_{21} = s_{32} = s_{13} = -1$, and

$$t_{ij} = \int_{\gamma_{ij}} \frac{dz}{z}. \quad (20)$$

The contour-integrals are performed along the curves $\gamma_{ij} = \{w_{ij}(s) | 0 \leq s \leq 1\}$, where $w_{ij}(s)$ is the unique (generally complex) root of the quadratic equation

$$(\mathbf{p}_j - \mathbf{p}_i)^2 z^2 + 2 \det(\mathbf{d} - \mathbf{p}_i, \mathbf{p}_j - \mathbf{p}_i) z + (\mathbf{d} - \mathbf{p}_i)^2 = s^2, \quad (21)$$

satisfying the condition

$$\text{Im} \frac{-(p_{jx} - p_{ix})z + d_y - p_{iy}}{(p_{jy} - p_{iy})z + d_x - p_{ix}} > 0. \quad (22)$$

The (complex) two-dimensional vector $\mathbf{d} = (d_x, d_y)$ is given by

$$\mathbf{d} = \frac{1}{\det(\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1)} \left[\frac{1}{2}(\mathbf{p}_3^2 - \mathbf{p}_1^2) + i(p_{30} - p_{10}) \right] (\mathbf{p}_2 - \mathbf{p}_1)_\perp + p_2 \leftrightarrow p_3. \quad (23)$$

The integration path $w_{ij}(s)$ can be written explicitly as²³

$$w_{ij}(s) = \frac{z_{ij}(s) - \bar{z}_{ij}}{|\mathbf{p}_j - \mathbf{p}_i|}. \quad (24)$$

Here $z_{ij}(s) = x_{ij}(s) + iy_{ij}(s)$ is a function of s with real and imaginary parts given by

$$x_{ij}(s) = \text{sgn}(p_{j0} - p_{i0}) \frac{1}{\sqrt{2}} \left[\sqrt{[a_{ij}(s)]^2 + (p_{j0} - p_{i0})^2} + a_{ij}(s) \right]^{1/2}, \quad (25)$$

$$y_{ij}(s) = -\frac{1}{\sqrt{2}} \left[\sqrt{[a_{ij}(s)]^2 + (p_{j0} - p_{i0})^2} - a_{ij}(s) \right]^{1/2}, \quad (26)$$

with

$$a_{ij}(s) = s^2 - \frac{1}{4}|\mathbf{p}_j - \mathbf{p}_i|^2 + \frac{(p_{j0} - p_{i0})^2}{|\mathbf{p}_j - \mathbf{p}_i|^2}. \quad (27)$$

The constant \bar{z}_{ij} is given by $\bar{z}_{ij} = \bar{x}_{ij} + i\bar{y}_{ij} = \bar{x}_{ijk} + i\bar{y}_{ijk}$, where k completes the index set $\{i, j\}$ to $\{i, j, k\} = \{1, 2, 3\}$, and

$$\bar{x}_{ijk} = \frac{|\mathbf{p}_j - \mathbf{p}_i|}{2\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} (\mathbf{p}_j - \mathbf{p}_k) \cdot (\mathbf{p}_k - \mathbf{p}_i), \quad (28)$$

$$\bar{y}_{ijk} = \frac{(\mathbf{p}_k - \mathbf{p}_i)(p_{j0} - p_{k0}) - (\mathbf{p}_j - \mathbf{p}_k)(p_{k0} - p_{i0})}{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} \cdot \frac{\mathbf{p}_j - \mathbf{p}_i}{|\mathbf{p}_j - \mathbf{p}_i|}. \quad (29)$$

One can easily show that $y_{ij}(s)$ increases strictly monotonically as a function of s , and $x_{ij}(s)$ increases (decreases) strictly monotonically if $\text{sgn}(p_{j0} - p_{i0}) > 0$ ($\text{sgn}(p_{j0} - p_{i0}) < 0$). The integration path $w_{ij}(s)$ thus has a simple shape. We finally note the following obvious symmetries under exchange of i and j :

$$x_{ji}(s) = -x_{ij}(s) \quad y_{ji}(s) = y_{ij}(s), \quad (30)$$

$$\bar{x}_{ji} = -\bar{x}_{ij} \quad \bar{y}_{ji} = -\bar{y}_{ij}. \quad (31)$$

V. 3-POINT LOOP IN COLLINEAR LOW-ENERGY LIMIT

We now derive the asymptotic behavior of the 3-point loop $I_3(p_1, p_2, p_3)$ in the collinear low-energy limit. To this end we substitute $p_{j0} \mapsto \lambda^3 p_{j0}$, $p_{jx} \mapsto \lambda^2 p_{jx}$, $p_{jy} \mapsto \lambda p_{jy}$ and expand I_3 as given by Eq. (19) in powers of λ .

We first expand the integration path $w_{ij}(s)$, Eq. (24). For the constants \bar{x}_{ijk} and \bar{y}_{ijk} one obtains

$$\bar{x}_{ijk} = \frac{|p_{jy} - p_{iy}|}{2\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} (p_{jy} - p_{ky})(p_{ky} - p_{iy}) + \mathcal{O}(\lambda), \quad (32)$$

$$\bar{y}_{ijk} = \lambda \frac{(p_{ky} - p_{iy})(p_{j0} - p_{k0}) - (p_{jy} - p_{ky})(p_{k0} - p_{i0})}{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} \frac{p_{jy} - p_{iy}}{|p_{jy} - p_{iy}|} + \mathcal{O}(\lambda^2). \quad (33)$$

For the functions $x_{ij}(s)$ and $y_{ij}(s)$ one finds

$$x_{ij}(s) = \text{sgn}(p_{j0} - p_{i0})s + \mathcal{O}(\lambda^2), \quad (34)$$

$$y_{ij}(s) = -\lambda^3 \frac{|p_{j0} - p_{i0}|}{2s} + \mathcal{O}(\lambda^5), \quad (35)$$

for $s > 0$, and

$$x_{ij}(0) = \lambda^2 \frac{p_{j0} - p_{i0}}{|p_{jy} - p_{iy}|} + \mathcal{O}(\lambda^3), \quad (36)$$

$$y_{ij}(0) = -\frac{\lambda}{2}|p_{jy} - p_{iy}| + \mathcal{O}(\lambda^2). \quad (37)$$

Inserting the expansion of the above auxiliary quantities into Eq. (24), and splitting the real and imaginary parts, one obtains

$$\begin{aligned} w_{ij}(0) = & -\frac{1}{\lambda} \frac{(p_{jy} - p_{ky})(p_{ky} - p_{iy})}{2\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} + \mathcal{O}(1) \\ & - i \left[\frac{1}{2} + \frac{(p_{ky} - p_{iy})(p_{j0} - p_{k0}) - (p_{jy} - p_{ky})(p_{k0} - p_{i0})}{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} \frac{p_{jy} - p_{iy}}{|p_{jy} - p_{iy}|^2} + \mathcal{O}(\lambda) \right], \end{aligned} \quad (38)$$

and

$$w_{ij}(s) = w_{ij}(0) + \frac{s}{\lambda} \frac{\text{sgn}(p_{j0} - p_{i0})}{|p_{jy} - p_{iy}|} + \mathcal{O}(1) + i \left[\frac{1}{2} + \mathcal{O}(\lambda) \right] \quad (39)$$

for $s > 0$.

The value of t_{ij} , Eq. (20), is given by the difference of natural logarithms at the end and the beginning of the integration path, plus contributions $\pm 2\pi i$ for each crossing of the branch cut on the negative real axis in the complex plane. For small λ one can write t_{ij} in a form where no case-dependent multiples of $2\pi i$ appear, namely³¹

$$t_{ij} = \ln[u_{ij}(s)] + \ln \left[1 + i \frac{y_{ij}(s)}{u_{ij}(s)} \right] \Big|_0^1, \quad (40)$$

where

$$u_{ij}(s) = x_{ij}(s) - \bar{x}_{ij} - i\bar{y}_{ij}. \quad (41)$$

The sum in Eq. (19) can be written as

$$I'_3 = \sum_{(i,j)=(1,2),(2,3),(3,1)} (t_{ij} - t_{ji}). \quad (42)$$

Forming the difference $t_{ij} - t_{ji}$, the first terms from Eq. (40) cancel due to the antisymmetry of $u_{ij}(s)$ in i and j , such that

$$t_{ij} - t_{ji} = \ln \left[1 + i \frac{y_{ij}(s)}{u_{ij}(s)} \right] - \ln \left[1 - i \frac{y_{ij}(s)}{u_{ij}(s)} \right] \Big|_0^1. \quad (43)$$

This is a suitable starting point for an expansion in powers of λ , since $y_{ij}(s)/u_{ij}(s)$ is of order λ for $s = 0$, and of order λ^3 for $s = 1$. Expanding the logarithm yields

$$t_{ij} - t_{ji} = \sum_{n=0}^{\infty} \frac{2}{2n+1} \left\{ \left[i \frac{y_{ij}(1)}{u_{ij}(1)} \right]^{2n+1} - \left[i \frac{y_{ij}(0)}{u_{ij}(0)} \right]^{2n+1} \right\}. \quad (44)$$

Inserting $u_{ij}(s)$ from Eq. (41) and expanding in powers of λ , one obtains

$$\begin{aligned} t_{ij} - t_{ji} = & 2i \frac{y_{ij}(0)}{\bar{x}_{ij}} + 2 \frac{\bar{y}_{ij} y_{ij}(0)}{\bar{x}_{ij}^2} \\ & + 2i \left[\frac{y_{ij}(1)}{x_{ij}(1) - \bar{x}_{ij}} + \frac{x_{ij}(0) y_{ij}(0)}{\bar{x}_{ij}^2} - \frac{\bar{y}_{ij}^2 y_{ij}(0)}{\bar{x}_{ij}^3} - \frac{y_{ij}^3(0)}{3\bar{x}_{ij}^3} \right] + \mathcal{O}(\lambda^4), \end{aligned} \quad (45)$$

where the first term is of order λ , the second of order λ^2 , and the third one of order λ^3 .

In the sum over pairs (i, j) , from Eq. (42), contributions of order λ and λ^2 to the single differences $t_{ij} - t_{ji}$ cancel, as do many terms of order λ^3 . To see this one has to insert expansions of the auxiliary quantities appearing in Eq. (45) in powers of λ , sometimes beyond the order presented in Eqs. (32) - (37). After a lengthy but straightforward calculation one obtains³¹

$$I'_3 = 2i \sum_{(i,j)=(1,2),(2,3),(3,1)} \frac{y_{ij}(1)}{x_{ij}(1) - \bar{x}_{ij}} + \mathcal{O}(\lambda^4). \quad (46)$$

Expanding \bar{x}_{ij} , $x_{ij}(1)$, and $y_{ij}(1)$ yields

$$\frac{y_{ij}(1)}{x_{ij}(1) - \bar{x}_{ij}} = -\frac{\lambda^3}{2} \frac{D_{ijk}(p_{j0} - p_{i0})}{D_{ijk} + \frac{1}{2} \text{sgn}(p_{j0} - p_{i0}) \text{sgn}(p_{jy} - p_{iy}) F_{ijk}} + \mathcal{O}(\lambda^4), \quad (47)$$

where

$$D_{ijk} = \det(\mathbf{p}_j - \mathbf{p}_k, \mathbf{p}_i - \mathbf{p}_k), \quad (48)$$

and

$$F_{ijk} = (p_{ky} - p_{jy})(p_{jy} - p_{iy})(p_{iy} - p_{ky}). \quad (49)$$

Note that D_{ijk} and F_{ijk} are both invariant under cyclic permutations of i, j , and k . Inserting Eq. (47) into Eq. (46), and dividing by $2\pi i \det(\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1)$, we obtain our final result for the collinear low-energy limit of the 3-point loop

$$\begin{aligned} \Pi_3(q_1, q_2, q_3) &= I_3(p_1, p_2, p_3) \\ &= \frac{1}{2\pi} \sum_{(i,j,k)=(1,2,3)+cyc.} \frac{p_{j0} - p_{i0}}{D_{123} + \frac{1}{2} \text{sgn}(p_{j0} - p_{i0}) \text{sgn}(p_{jy} - p_{iy}) F_{123}} + \mathcal{O}(\lambda). \end{aligned} \quad (50)$$

The 3-point loop is thus generically finite for $\lambda \rightarrow 0$. The limit is real and depends on the ratios $(p_{j0} - p_{i0})/D_{123}$ and $(p_{j0} - p_{i0})/F_{123}$. Note that $|F_{123}| = q_{1y}q_{2y}q_{3y}$, while $|D_{123}|$ is twice the area of the triangle with corners \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 , or, equivalently, of the triangle obtained by attaching the vectors \mathbf{q}_1 , \mathbf{q}_2 and \mathbf{q}_3 to each other. $\Pi_3(q_1, q_2, q_3)$ vanishes if frequency variables q_{j0} are set to zero before scaling \mathbf{q}_j to zero, in agreement with the result of Hertz and Klenin.²⁶ It also vanishes if either $\text{sgn}(p_{j0} - p_{i0}) = \text{sgn}(p_{jy} - p_{iy})$ for all (i, j) or $\text{sgn}(p_{j0} - p_{i0}) = -\text{sgn}(p_{jy} - p_{iy})$ for all (i, j) . In these cases the contributions to the sum over cyclic permutations of $(1, 2, 3)$ in Eq. (50) cancel.

The expression on the right hand side of Eq. (50) is invariant under permutations of q_1 , q_2 , and q_3 , so that it also describes the collinear low-energy limit of the symmetrized 3-point loop $\Pi_3^S(q_1, q_2, q_3)$.

VI. N-POINT LOOP IN COLLINEAR LOW-ENERGY LIMIT

The reduction formula (17) relates the N -point loop to a linear combination of 3-point loops. The coefficients are determined by the quantities $f_{i\nu}(\mathbf{d}^{ijk})$ defined in Eq. (18). In the collinear low-energy limit, the latter become frequency independent and real, and they scale as

$$\begin{aligned} f_{i\nu}(\mathbf{d}^{ijk}) &= \frac{\lambda^2}{2}(p_{iy}^2 - p_{\nu y}^2) \\ &+ \frac{\lambda^2}{2} \left[(p_{ky}^2 - p_{iy}^2) \frac{D_{ij\nu}}{D_{ijk}} + j \leftrightarrow k \right] + \mathcal{O}(\lambda^3). \end{aligned} \quad (51)$$

Inserting this and Eq. (50) for I_3 into Eq. (17), one obtains an explicit formula for the N -point loop in the collinear low-energy limit. It is remarkable that $\Pi_N(q_1, \dots, q_N) = I_N(p_1, \dots, p_N)$ is a *rational* function of all momenta and frequencies in this limit. For $N > 3$, it *diverges* as

$$\Pi_N \propto \lambda^{2(3-N)} \quad (52)$$

for $\lambda \rightarrow 0$.

The degree of divergence of Π_N is not reduced upon symmetrization, so that the symmetrized N -point loop also diverges as

$$\Pi_N^S \propto \lambda^{2(3-N)}. \quad (53)$$

We have confirmed the absence of significant cancellations by computing the scaling behavior of Π_N^S for various choices of N and q_1, \dots, q_N . This result is remarkable since strong and systematic cancellations have been shown to occur upon symmetrization when the limit $q_j \rightarrow 0$ is taken more conventionally. In particular, uniformly scaled N -point loops $\Pi_N(\lambda q_1, \dots, \lambda q_N)$ diverge as λ^{2-N} , while their symmetrized counterparts $\Pi_N^S(\lambda q_1, \dots, \lambda q_N)$ remain finite.^{23,24} The arguments establishing the cancellation of divergences in the uniform small- q limit do not apply in the collinear low-energy limit.

The divergence in Eq. (53) is the “worst case scenario” compatible with simple power counting: In the collinear low-energy limit the integration measure in the definition of the N -point loop scales as λ^6 , while each of the N propagators diverges as λ^{-2} , such that the loop may diverge as λ^{6-2N} (but not stronger). What we have shown is that this divergence is neither reduced by oscillations of the integrand under the k -integral in Eq. (14), nor by cancellations in the sum over permutations contributing to the symmetrized loops. Note that the power-counting does not change if the bare propagator $G_0(k)$ in the N -point loops is replaced by a propagator $G(k)$ with a self-energy proportional to $k_0^{2/3}$, since this interacting propagator also diverges as λ^{-2} .

VII. POWER COUNTING OF ORDER PARAMETER INTERACTION

Now that we have determined the scaling behavior of the N -point loop in the collinear low-energy limit, we can assess the relevance of the N -point order parameter interactions in the effective action $S[\phi]$ by using standard power counting. To see how the interaction terms evolve compared to the quadratic part of the action, we rescale the field ϕ in such a way that the bare critical action $S_0[\phi]$, Eq. (9), remains invariant. Since the integration measure scales as λ^6 , and the inverse bare propagator $D_0^{-1}(q)$ as λ^2 , we have to rescale the field as $\phi \mapsto \lambda^{-4}\phi$. The N -point interaction terms in $S[\phi]$ are composed of N energy-momentum integrals, a delta-function for energy-momentum conservation, the N -point loop, and a product of N fields $\phi(q_1) \dots \phi(q_N)$. The N -point interaction therefore scales as

$$S_{I,N}[\phi] \propto \lambda^{6N} \lambda^{-6} \lambda^{2(3-N)} \lambda^{-4N} = \lambda^0. \quad (54)$$

All N -point interactions contributing to $S[\phi]$ are thus *marginal* in the collinear low-energy scaling limit. Hence, the effective order parameter action cannot be truncated at any fi-

nite order. At least such a truncation is not justified by power counting. Furthermore, the interaction terms have singular momentum and energy dependences, which cannot be represented by local interactions.

Whether the Gaussian fixed point remains stable or not depends therefore entirely on the behavior of fluctuation contributions. To get an idea one might compute low order fluctuation corrections to $S_{I,3}[\phi]$ and $S_{I,4}[\phi]$. Metlitski and Sachdev²⁰ have shown that the bare Gaussian propagator $D_0(q)$ does not receive qualitative modifications up to three-loop order in the coupled fermion-boson theory underlying the effective action $S[\phi]$.^{21,22} Obtaining a general conclusion on the fate of the Gaussian fixed point seems difficult, however, since the theory has no obvious expansion parameter.

Let us also discuss the power counting for a generalized Gaussian part of the effective action, of the form $S_0[\phi] = \frac{1}{2} \int_q (\chi \mathbf{q}^{1+\epsilon} + \gamma \frac{q_0}{|\mathbf{q}|}) \phi(q) \phi(-q)$, where $\epsilon \in [0, 1]$. This generalization was introduced by Nayak and Wilczek³² for the sake of a controlled expansion in ϵ . It was recently used by Mross et al.³³ for the purpose of defining a manageable large- N_f limit of the theory, where N_f is the number of fermion flavors. The case $\epsilon = 0$ is related to the theory of electrons in a half-filled Landau level,³⁴ while $\epsilon = 1$ describes the nematic QCP and related systems as discussed above. For arbitrary ϵ , the scaling limit (12) has to be generalized to $q_{j0} \mapsto \lambda^{2+\epsilon} q_{j0}$, $q_{jx} \mapsto \lambda^2 q_{jx}$, and $q_{jy} \mapsto \lambda q_{jy}$, corresponding to a dynamical exponent $z = 2 + \epsilon$. The limit is still collinear and “static” (in the sense that frequencies scale to zero faster than the modulus of momenta). In light of the results for $\epsilon = 1$ it is thus likely that the degree of divergence of the N -point loops is not reduced by cancellations in this scaling limit for $\epsilon < 1$, too, such that it can be estimated by naive power counting. One then obtains $\Pi_N^S \propto \lambda^{5+\epsilon-2N}$. Powercounting for the effective action then yields $S_{I,N}[\phi] \propto \lambda^0$ as in the special case $\epsilon = 1$. The N -point interactions are thus marginal in the collinear low-energy limit, for any N and ϵ . The size of ϵ does not matter here.

We finally compare to the situation in three dimensions. Here finite ϕ^N interactions are even more irrelevant than in two dimensions, but one may again wonder about the special role of momentum transfers tangential to the Fermi surface in a certain Fermi point (and its antipode), which are mutually *coplanar*. Considering the coplanar low-energy scaling limit $q_0 \mapsto \lambda^3 q_0$, $q_x \mapsto \lambda^2 q_x$, and $q_{y,z} \mapsto \lambda q_{y,z}$, one expects a divergence $\Pi_N^S \propto \lambda^{7-2N}$ for $N \geq 4$. These divergences are however not strong enough to make $S_{I,N}[\phi]$ marginal or relevant. Power counting yields $S_{I,3}[\phi] \propto \lambda^{1/2}$, and $S_{I,N}[\phi] \propto \lambda^{N/2}$ for $N \geq 4$. The Gaussian fixed

point is thus clearly stable in three dimensions.

VIII. CONCLUSION

In summary, we have analyzed the scaling behavior of N -point interactions between order parameter fluctuations at a nematic QCP and for other quantum critical electron systems with a scalar order parameter in two dimensions. The N -point interactions are given by symmetrized fermionic loops with N vertices. We have shown that these loops exhibit a singular momentum and energy dependence for all $N \geq 3$, so that they cannot be represented by a local interaction. For $N \geq 4$, they diverge in the collinear low-energy limit, where energy variables scale to zero faster than momenta, and momenta become increasingly collinear. We have derived explicit expressions for the momentum and energy dependences in that limit. The degree of divergence is not reduced by any cancellations. From standard power counting one then obtains that all N -point interactions are marginal.

The effective action is thus dominated by interactions between fluctuations with collinear momenta, as noted already previously.²⁰ It cannot be truncated at any finite order and none of the N -point terms can be represented by a local interaction. In particular, approximating the interactions by a local quartic term, as in Hertz' theory, is inadequate. The same conclusion can be drawn for the effective action describing fermions coupled to a $U(1)$ gauge field.

Marginality of all N -point order parameter interactions has already been obtained for a spin density wave QCP in two dimensions.¹¹ In that case the fluctuation propagator is strongly renormalized by these interactions. In particular, anomalous scaling dimensions appear.^{10,12} For the nematic QCP and related theories it is presently unclear whether the Gaussian fixed point remains stable. Perturbative calculations have not yet revealed any singular renormalization of the bare Gaussian propagator.^{19,20}

Acknowledgments

We are grateful to D. Belitz, A. Chubukov, M. Metlitski, and S. Sachdev for valuable discussions, and to N. Hasselmann for a critical reading of the manuscript.

-
- * Present address: Institut für Physik, Johannes-Gutenberg-Universität, 55099 Mainz, Germany
 - ¹ S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, UK, 1999).
 - ² M. Vojta, Rep. Prog. Phys. **66**, 2069 (2003).
 - ³ H. v. Löhneysen, A. Rosch, M. Vojta, and P. Wölfle, Rev. Mod. Phys. **79**, 1015 (2007).
 - ⁴ J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).
 - ⁵ A. J. Millis, Phys. Rev. B **48**, 7183 (1993).
 - ⁶ D. Belitz, T. R. Kirkpatrick, and T. Vojta, Rev. Mod. Phys. **77**, 579 (2005).
 - ⁷ T. R. Kirkpatrick and D. Belitz, Phys. Rev. B **53**, 14364 (1996); see also D. Belitz, T. R. Kirkpatrick, M. T. Mercaldo, and S. L. Sessions, Phys. Rev. B **63**, 174427 (2001); *ibid* 174428 (2001).
 - ⁸ D. Belitz, T. R. Kirkpatrick, and T. Vojta, Phys. Rev. B **55**, 9452 (1997).
 - ⁹ D. Belitz, T. R. Kirkpatrick, and T. Vojta, Phys. Rev. Lett. **82**, 4707 (1999).
 - ¹⁰ A. Abanov, A. V. Chubukov, and J. Schmalian, Adv. Phys. **52**, 119 (2003).
 - ¹¹ A. Abanov and A. V. Chubukov, Phys. Rev. Lett. **93**, 255702 (2004)
 - ¹² M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075128 (2010).
 - ¹³ E. Fradkin, S. A. Kivelson, M. J. Lawler, J. P. Eisenstein, and A. P. Mackenzie, Annu. Rev. Condens. Matter Phys. **1**, 153 (2010).
 - ¹⁴ W. Metzner, C. Castellani, and C. Di Castro, Adv. Phys. **47**, 317 (1998).
 - ¹⁵ P. A. Lee, N. Nagaosa, and X. G. Wen, Rev. Mod. Phys. **78**, 17 (2006).
 - ¹⁶ V. Oganesyan, S. A. Kivelson, and E. Fradkin, Phys. Rev. B **64**, 195109 (2001).
 - ¹⁷ W. Metzner, D. Rohe, and S. Andergassen, Phys. Rev. Lett. **91**, 066402 (2003).
 - ¹⁸ P. A. Lee, Phys. Rev. Lett. **63**, 680 (1989).
 - ¹⁹ Y. B. Kim, A. Furusaki, X. G. Wen, and P. A. Lee, Phys. Rev. B **50**, 17917 (1994).
 - ²⁰ M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010).
 - ²¹ The \mathbf{q}^2 -term survives to three-loop order and the dynamical exponent remains $z = 3$. However,

in a generalization to an arbitrary number of fermion flavors N_f , the prefactor of the \mathbf{q}^2 -term acquires a dependence on N_f which is very different from the RPA result.

- ²² The frequency dependence has not yet been computed to three-loop order. The Landau damping term might be modified to an expression of the form $\mathbf{q}^2 f(q_0/|\mathbf{q}|^3)$, with a non-linear function f (M. A. Metlitski, private communication).
- ²³ A. Neumayr and W. Metzner, Phys. Rev. B **58**, 15449 (1998).
- ²⁴ C. Kopper and J. Magnen, Ann. Henri Poincaré **2**, 513 (2001).
- ²⁵ L. Dell’Anna and W. Metzner, Phys. Rev. B **73**, 045127 (2006).
- ²⁶ J. A. Hertz and M. A. Klenin, Phys. Rev. B **10**, 1084 (1974).
- ²⁷ J. Polchinski, Nucl. Phys. B **422**, 617 (1994).
- ²⁸ B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B **50**, 14048 (1994).
- ²⁹ J. Feldman, H. Knörrer, R. Sinclair, and E. Trubowitz, in *Singularities*, edited by G. M. Greuel (Birkhaeuser, Basel, 1998).
- ³⁰ For an elementary proof, see A. Neumayr and W. Metzner, J. Stat. Phys. **96**, 613 (1999).
- ³¹ S. Thier, Diploma thesis, University of Stuttgart (2011).
- ³² C. Nayak and F. Wilczek, Nucl. Phys. B **417**, 359 (1994); **430**, 534 (1994).
- ³³ D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, Phys. Rev. B **82**, 045121 (2010).
- ³⁴ B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).